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# A Note on Polynomial Interpolation at the Chebyshev Extrema Nodes

L. BRUTMAN

*Department of Computer Science, University of Haifa, Haifa, Israel*

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Some new properties of the Lebesgue function associated with interpolation at the Chebyshev extrema nodes are established. By estimating the degree of asymmetry of the Lebesgue function in the interval of interest, the estimate of Ehlich and Zeller for the norm of the corresponding interpolation operator is improved.


## I. INTRODUCTION

Let $X = \{x_k\}_{k=0}^{m}$ be a given set of $(m+1)$ distinct points in $[-1, 1]$ and denote by $C[-1, 1]$ the Banach space of continuous functions on $[-1, 1]$ equipped with the uniform norm. To each $f(x) \in C[-1, 1]$ there corresponds a unique interpolation polynomial $P_m(X; x)$ of degree at most $m$ which can be expressed in the Lagrangian form:

$$P_m(X; x) = \sum_{k=0}^{m} f(x_k)\, l_k(X; x), \tag{1}$$

where

$$l_k(X; x) = \prod_{\substack{i=0 \\ i \neq k}}^{m} (x - x_i)/(x_k - x_i).$$

In interpolation theory the behavior of the function

$$\lambda_m(X; x) = \sum_{k=0}^{m} |l_k(X; x)|, \tag{2}$$

usually referred to as the Lebesgue function, is of great importance since the



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operator norm of $P_m(X; \cdot)$ (as an operator on $C[-1, 1]$) equals the supnorm of its Lebesgue function

$$\|P_m(X)\| = \max_{-1 \leqslant x \leqslant 1} \lambda_m(X; x). \tag{3}$$

Recently, stimulated by the proof of the Bernstein and the Erdös conjectures concerning optimal choice of interpolation nodes [1, 5], special interest has been devoted to the analysis of the behavior of the Lebesgue function for specific sets of interpolation nodes. The properties of the Lebesgue function corresponding to the set of Chebyshev roots $T = \{\cos[(2k+1)\pi/(m+1)]\}_{k=0}^{m}$ were thoroughly investigated in [2]. It has been also shown that the set of nodes $U = \{\cos(k\pi/m)\}_{k=0}^{m}$, coinciding with the Chebyshev extrema, failed to be a good approximation to the optimal interpolation set. Nevertheless, this set of nodes is of considerable interest since as was established by Ehlich and Zeller [3] (see also McCabe and Phillips [6]), the norm of the corresponding interpolation operator $P_n(U)$ is less than the norm of the operator $P_n(T)$ induced by interpolation at the Chebyshev roots. Namely, the following relation holds:

$$\begin{aligned} \|P_m(U)\| &= \|P_{m-1}(T)\|, & m &= 1, 3, 5, \ldots, \\ &= \|P_{m-1}(T)\| - \alpha_m, \; 0 < \alpha_m < \frac{1}{m^2}, & m &= 2, 4, 6, \ldots. \end{aligned} \tag{4}$$

Estimate (4) appears also in Rivlin [7, 8], where the author expressed interest in "closer scrutiny of the quantities $\alpha_m$."

In the present work we exploit the convenient representation of the Lebesgue function $\lambda_m(U; x)$ obtained in [2] and by proving some new properties of this function, establish a sharper bound for $\alpha_m$. As a by-product, we obtain two trigonometric identities, the proof of which is given in the Appendix.

## II. Results

Let $l_k(U; x) \equiv l_k(x)$ (In the following we deal mainly with the $U$-set of nodes and when no ambiguity arises the symbol $U$ will be omitted.) be the fundamental polynomials induced by interpolation at the set of Chebyshev extrema $U = \{x_k = \cos(k\pi/m)\}_{k=0}^{m}$. We use the standard trigonometric transformation $x = \cos\theta$ ($x_k = \cos\theta_k$, $\theta_k = k\pi/m$) and denote by $\lambda_m(U; \theta) \equiv \lambda_m(\theta)$ the corresponding Lebesgue function

$$\lambda_m(\theta) = \sum_{k=0}^{m} |l_k(\cos\theta)|. \tag{5}$$

By introducing a new variable $\varphi = \theta - \theta_{j-1}$, $\varphi \in (0, \pi/m)$ and denoting the restriction of the Lebesgue function $\lambda_m(\theta)$ to the subinterval $I_j = (\theta_{j-1}, \theta_j)$, $j = 1, 2, \ldots, m$ by $\lambda_m^j(\theta)$, one can use the following convenient representation obtained in [2]:

$$\lambda_m^j(\theta) \equiv \lambda_m(I_j, \varphi) = \frac{\sin(m\varphi)}{m} \left\{ \sum_{k=0}^{2j-2} \frac{1}{\sin(\varphi + \theta_k)} - \sum_{k=2j-1}^{m-1} \cot(\varphi + \theta_k) \right\}. \quad (6)$$

From now on we deal with the even case where $m = 2n$. Since $\|P_{2n}(U)\| = \max_{0 \leqslant \varphi \leqslant \pi/2n} \lambda_{2n}(I_n, \varphi)$ (see [2]), we concentrate our attention on the analysis of the function $\lambda_{2n}(I_n, \varphi)$ denoted further by $F_{2n}(\varphi)$. It follows from (6) that

$$F_{2n}(\varphi) = \frac{\sin(2n\varphi)}{2n} \left\{ \sum_{k=0}^{2n-2} \frac{1}{\sin(\varphi + \theta_k)} - \cot(\varphi + \theta_{2n-1}) \right\}, \quad (7)$$

where $\theta_k = k\pi/(2n)$, $k = 0, 1, \ldots, 2n$.

The idea of estimating the "degree of asymmetry" of this function is of central importance in our analysis. Let $A_{2n}(\varphi) = F_{2n}[(\pi/2n) - \varphi] - F_{2n}(\varphi)$, $0 \leqslant \varphi \leqslant \pi/(4n)$. The maximal value of this difference will be used as a measure of asymmetry of the Lebesgue function. The following estimate holds:

THEOREM 1.

$$\max_{0 \leqslant \varphi \leqslant \pi/4n} A_{2n}(\varphi) < \frac{\pi}{16n} \tan\left(\frac{\pi}{4n}\right).$$

*Proof.* First we rewrite (7) in the form

$$F_{2n}(\varphi) = \frac{\sin(2n\varphi)}{2n} \left\{ \sum_{k=0}^{2n-1} \frac{1}{\sin(\varphi + \theta_k)} - \tan\left(\frac{\pi}{4n} - \frac{\varphi}{2}\right) \right\}. \quad (8)$$

Now let $G_{2n}(\varphi) = \sum_{k=0}^{2n-1} [1/\sin(\varphi + \theta_k)]$. It is clear that $G_{2n}[(\pi/2n) - \varphi] = G_{2n}(\varphi)$, $0 \leqslant \varphi \leqslant \pi/(4n)$, and hence

$$A_{2n}(\varphi) = \frac{\sin(2n\varphi)}{2n} \left\{ \tan\left(\frac{\pi}{4n} - \frac{\varphi}{2}\right) - \tan\left(\frac{\varphi}{2}\right) \right\} \equiv \frac{\sin(2n\varphi)}{2n} H_n(\varphi). \quad (9)$$

Since

$$H_n'(\varphi) = -\frac{1}{2} \left\{ 1/\cos^2\left(\frac{\pi}{4n} - \frac{\varphi}{2}\right) + 1/\cos^2\left(\frac{\varphi}{2}\right) \right\} < 0, \quad (10)$$

the function $H_n(\varphi)$ is monotone decreasing. Furthermore, an easy computation reveals

$$H_n''(\varphi)=\frac{1}{2}\left\{\sin\left(\frac{\pi}{4n}-\frac{\varphi}{2}\right)\Big/\cos^3\left(\frac{\pi}{4n}-\frac{\varphi}{2}\right)-\sin\left(\frac{\varphi}{2}\right)\Big/\cos^3\left(\frac{\varphi}{2}\right)\right\}>0, \tag{11}$$

showing that $H_n(\varphi)$ is also a convex function on $(0, \pi/4n)$. Thus,

$$H_n(\varphi)<\tan\left(\frac{\pi}{4n}\right)-\left[\tan\left(\frac{\pi}{4n}\right)\Big/\frac{\pi}{4n}\right]\varphi \tag{12}$$

and, therefore,

$$A_{2n}(\varphi)\leqslant\varphi\cdot\tan\left(\frac{\pi}{4n}\right)\left(1-\frac{4n}{\pi}\varphi\right). \tag{13}$$

It remains to note that the function on the right-hand side of (13) attains its maximal value at $\varphi=\pi/(8n)$. This concludes the proof of the theorem.

We proceed now to prove the following property:

THEOREM 2. *The Lebesgue function $\lambda_{2n}(I_n, \varphi)$ is a concave function of $\varphi$ on $[0, \pi/(2n)]$.*

*Proof.* For convenience we return to the variable $\theta$ (related in our case to the variable $\varphi$ as follows: $\theta=\varphi+\theta_{n-1}$) and prove an equivalent statement that $\lambda_{2n}(\theta)$ is a concave function of $\theta$ on $[\theta_{n-1}, \theta_n]$. We start by observing that the function $\lambda_{2n}^n(\theta)$ coinciding with $\lambda_{2n}(\theta)$ for $\theta\in[\theta_{n-1}, \theta_n]$ is an even trigonometric polynomial of degree $\leqslant 2n$. Let us denote this polynomial by $\tau_{2n}(\theta)$ and consider its behaviour for $0\leqslant\theta\leqslant\pi$. It is clear from the definition of the Lebesgue function that

$$\begin{aligned}\tau_{2n}(\theta_k)&=(-1)^{n+k-1}, && 0\leqslant k<n-1,\\ &=1, && k=n-1, n\\ &=(-1)^{n+k}, && n<k\leqslant 2n-1.\end{aligned} \tag{14}$$

Hence $\tau_{2n}(\theta)$ has at least $(2n-1)$ sign changes in $(0, \pi)$. Thus its first derivative $\tau_{2n}'(\theta)$, being an odd trigonometric polynomial has not less than $(4n-2)$ zeros in $[0, 2\pi)$. Further the second derivative $\tau_{2n}''(\theta)$ also has at least $(4n-2)$ zeros in $[0, 2\pi)$. On the other hand $\tau_{2n}''(\theta)$, as a trigonometric polynomial of degree $\leqslant 2n$, has at most $4n$ zeros in $[0, 2\pi)$. Therefore, between any two successive extrema of $\tau_{2n}(\theta)$ there is exactly one sign change of $\tau''_{2n}(\theta)$. Hence, it suffices to verify that

$$\tau''_{2n}(\theta_n) < 0, \tag{15}$$

$$\tau''_{2n}(\theta_{n-1}) < 0, \tag{16}$$

in order to establish the concavity of $\lambda_{2n}(\theta)$ on $[\theta_{n-1}, \theta_n]$.

To prove (15) we use the representation of the fundamental polynomials in the following form [6]:

$$l_k(\cos\theta) \equiv p_k(\theta) = \frac{\varepsilon_k}{n} \sum_{r=0}^{2n}{}'' \cos(r\theta_k) \cdot \cos(r\theta), \qquad k = 0, 1, \dots, 2n, \tag{17}$$

where $\sum''$ denotes a sum whose first and last terms are halved, and

$$\begin{aligned} \varepsilon_k &= \tfrac{1}{2}, \qquad k = 0, 2n, \\ &= 1, \qquad k = 1, 2, \dots, (2n-1). \end{aligned}$$

Thus,

$$p''_k(\theta_n) = p''_k(\pi/2) = -\frac{\varepsilon_k}{n} \sum_{r=0}^{2n}{}'' r^2 \cos(r\theta_k) \cos\left(\frac{r\pi}{2}\right). \tag{18}$$

One can easily check that $p''_{n-k}(\theta_n) = p''_{n+k}(\theta_n)$, $k = 1, 2, \dots, n$, and since for $\theta \in [\theta_{n-1}, \theta_n]$,

$$\tau''_{2n}(\theta) = p''_n(\theta) + \sum_{k=1}^{n} (-1)^{k-1} [p''_{n-k}(\theta) - p''_{n+k}(\theta)], \tag{19}$$

we obtain

$$\begin{aligned} \tau''_{2n}(\theta_n) &= -\frac{1}{n} \sum_{r=0}^{2n}{}'' r^2 \cos^2\left(\frac{r\pi}{2}\right) \\ &= -\frac{1}{n} \left\{ 2^2 + 4^2 + \cdots + (2n-2)^2 + \frac{1}{2}(2n)^2 \right\} = -\frac{4n^2+2}{3}, \end{aligned} \tag{20}$$

completing the proof of (15).

Verification of (16) is more complicated and based on two trigonometric identities the proof of which is given in Lemmas 1 and 2 in the Appendix. It follows from (17) that

$$p_k''(\theta_{n-1}) = p_k''\left(\frac{\pi}{2} - \frac{\pi}{2n}\right)$$

$$= -\frac{\varepsilon_k}{n} \sum_{r=0}^{2n}{}'' r^2 \cos(r\theta_k) \cdot \cos\left(\frac{r\pi}{2}\right) \cdot \cos\left(\frac{r\pi}{2n}\right)$$

$$-\frac{\varepsilon_k}{n} \sum_{r=0}^{2n}{}'' r^2 \cos(r\theta_k) \cdot \sin\left(\frac{r\pi}{2}\right) \cdot \sin\left(\frac{r\pi}{2n}\right) \equiv a_k + b_k, \quad (21)$$

and in accordance with (19) we deduce

$$\tau_{2n}''(\theta_{n-1}) = a_n + \sum_{k=1}^{n} (-1)^{k-1}(a_{n-k} - a_{n+k}) + b_n$$

$$+ \sum_{k=1}^{n} (-1)^{k-1}(b_{n-k} - b_{n+k}). \quad (22)$$

Now, since $a_{n+k} = a_{n-k}$ ($k = 1, 2,..., n$), the first sum in (22) cancels, while by applying Lemma 1 we derive

$$a_n = -\frac{1}{n} \sum_{r=0}^{2n}{}'' r^2 \cos^2\left(\frac{r\pi}{2}\right) \cos\left(\frac{r\pi}{2n}\right)$$

$$= -\frac{4}{n}\left\{\frac{n^2}{2} + \sum_{s=1}^{n} s^2 \cos\left(\frac{s\pi}{n}\right)\right\} = 2/\sin^2\left(\frac{\pi}{2n}\right). \quad (23)$$

Furthermore, one can easily verify that $b_n = 0$ and $b_{n+k} = -b_{n-k}$ ($k = 1, 2,..., n$). Hence

$$b_n + \sum_{k=1}^{n} (-1)^k [b_{n-k} - b_{n+k}]$$

$$= 2 \sum_{k=0}^{n-1} (-1)^{n-1+k} b_k$$

$$= \frac{2(-1)^n}{n} \sum_{s=1}^{n} (-1)^{s+1}(2s-1)^2 \sin\left[\frac{(2s-1)\pi}{2n}\right]$$

$$\times \sum_{k=0}^{n-1}{}' (-1)^k \cos\left[\frac{k(2s-1)\pi}{2n}\right], \quad (24)$$

where $\sum'$ denotes a sum whose first term is halved. Now, by employing the identity

$$\sum_{k=0}^{n-1} (-1)^k \cos\left[\frac{k(2s-1)\pi}{2n}\right] = \frac{(-1)^{n+s}}{2} \tan\left[\frac{(2s-1)\pi}{4n}\right], \quad (25)$$

we obtain after some simplification

$$2 \sum_{k=0}^{n-1} (-1)^{n-1+k} b_k = -\frac{2}{n} \sum_{s=1}^{n} (2s-1)^2 \sin\left[\frac{(2s-1)\pi}{4n}\right], \tag{26}$$

which implies, in view of Lemma 2,

$$2 \sum_{k=0}^{n-1} (-1)^{n-1+k} b_k = -\frac{1}{n} \left\{\frac{n(4n^2-1)}{3} + \frac{2n\cos(\pi/2n)}{\sin^2(\pi/2n)}\right\}. \tag{27}$$

Combining (22), (23), (24), and (27) yields

$$\tau_{2n}(\theta_{n-1}) = -\frac{4n^2-1}{3} + \frac{1}{\cos^2(\pi/4n)} < 0. \tag{28}$$

This completes the proof of the theorem.

We are now in a position to derive the following improvement of (4):

THEOREM 3. *For any* $n = 1, 2, \ldots,$

$$\|P_{2n}(U)\| = \|P_{2n-1}(T)\| - \alpha_{2n}$$

*with*

$$\frac{\pi/8}{(2n)^2} < \alpha_{2n} < \frac{2(\sqrt{2}-1)}{(2n)^2}.$$

*Proof.* It follows from (7) that

$$\begin{aligned} F_{2n}\left(\frac{\pi}{4n}\right) &= \frac{1}{2n}\cot\left(\frac{\pi}{4n}\right) + \frac{1}{2n}\sum_{k=1}^{2n-1} 1/\sin\left[\frac{(2k-1)\pi}{4n}\right] \\ &= \frac{1}{2n}\sum_{k=1}^{2n} 1/\sin\left[\frac{(2k-1)\pi}{4n}\right] - \frac{1}{2n}\tan\left(\frac{\pi}{8n}\right) \\ &= \frac{1}{2n}\sum_{k=1}^{2n} \cot\left[\frac{(2k-1)\pi}{8n}\right] - \frac{1}{2n}\tan\left(\frac{\pi}{8n}\right). \end{aligned} \tag{29}$$

On the other hand, it is known (see, e.g., [8]) that

$$\frac{1}{2n}\sum_{k=1}^{2n} \cot\left[\frac{(2k-1)\pi}{8n}\right] = \|P_{2n-1}(T)\| \tag{30}$$

and hence

$$F_{2n}\left(\frac{\pi}{4n}\right) = \|P_{2n-1}(T)\| - \frac{1}{2n}\tan\left(\frac{\pi}{8n}\right). \tag{31}$$

Now by applying Theorem 2 we find

$$\|P_{2n}(U)\| = \max_{0 \leqslant \varphi \leqslant \pi/4n} F_{2n}(\varphi) \leqslant F_{2n}\left(\frac{\pi}{4n}\right) + \frac{1}{2} \max_{0 \leqslant \varphi \leqslant \pi/4n} A_{2n}(\varphi), \tag{32}$$

and in conjunction with Theorem 1 and (31) this leads to the estimate

$$\|P_{2n-1}(T)\| - \frac{1}{2n} \tan\left(\frac{\pi}{8n}\right)$$
$$< \|P_{2n}(U)\| < \|P_{2n-1}(T)\| - \left\{\frac{1}{2n} \tan\left(\frac{\pi}{8n}\right) - \frac{\pi}{32n} \tan\left(\frac{\pi}{4n}\right)\right\}. \tag{33}$$

It remains to note that for $n = 1, 2, \ldots,$

$$\frac{\pi}{8n} < \tan\left(\frac{\pi}{8n}\right) \leqslant \tan\left(\frac{\pi}{8}\right)\Big/ n; \qquad \tan\left(\frac{\pi}{4n}\right) \leqslant \frac{1}{n}. \tag{34}$$

The theorem follows.

## Appendix

LEMMA 1.

$$\sum_{k=1}^{n} k^2 \cos\left(\frac{k\pi}{n}\right) = -\frac{n^2}{2} - n/2 \sin^2\left(\frac{\pi}{2n}\right), \qquad n = 1, 2, \ldots.$$

*Proof.* We apply the well-known Abel transformation (see, e.g., [9, Vol. I, p. 11]) and take into account that

$$\sum_{s=1}^{k} \cos(s\alpha) = -\frac{1}{2} + \sin\left[\frac{(2k+1)\alpha}{2}\right]\Big/ 2 \sin\left(\frac{\alpha}{2}\right) \tag{35}$$

in order to obtain

$$\sum_{k=1}^{n} k^2 \cos\left(\frac{k\pi}{n}\right) = -\frac{1}{2}(n^2+1) - \frac{1}{2\sin(\pi/2n)} \sum_{k=1}^{n-1} (2k+1) \sin\left[\frac{(2k+1)\pi}{2n}\right]. \tag{36}$$

Let $B_{n-1} = \sum_{k=1}^{n-1} (2k+1) \sin[(2k+1)\pi/(2n)]$. Applying the Abel transformation again and using the trigonometric identity [4]

$$\sum_{s=1}^{k} \sin\left[\frac{(2s-1)\pi}{2n}\right] = \sin^2\left(\frac{k\pi}{2n}\right)\Big/ \sin\left(\frac{\pi}{2n}\right) \tag{37}$$

we find

$$B_{n-1} = -\sin\left(\frac{\pi}{2n}\right) + \frac{2n-1}{\sin(\pi/2n)} - \frac{2}{\sin(\pi/2n)} \sum_{k=1}^{n-1} \sin^2\left(\frac{k\pi}{2n}\right). \tag{38}$$

Now, since $\sum_{k=1}^{n-1} \sin^2[k\pi/2n] = (n-1)/2$ we deduce from (38) that

$$B_{n-1} = -\sin\left(\frac{\pi}{2n}\right) + n/\sin\left(\frac{\pi}{2n}\right). \tag{39}$$

Substitution of (39) in (36) yields the desired result.

LEMMA 2.

$$\sum_{s-1}^{n} (2s-1)^2 \sin^2\left[\frac{(2s-1)\pi}{4n}\right] = \frac{1}{6} n(4n^2-1) + \frac{n \cdot \cos(\pi/2n)}{\sin^2(\pi/2n)}. \qquad n = 1, 2, \ldots.$$

*Proof.* By the Abel transformation and the following easily verified trigonometric identity

$$\sum_{s=1}^{k} \sin^2\left[\frac{(2s-1)\pi}{4n}\right] = \frac{k}{2} - \frac{\sin(k\pi/n)}{4\sin(\pi/2n)} \tag{40}$$

we derive

$$\sum_{s=1}^{n} (2s-1)^2 \sin^2\left[\frac{(2s-1)\pi}{4n}\right] = \frac{1}{2} n(2n-1)^2 - 4\sum_{k=1}^{n-1} k^2 + \frac{2\sum_{k=1}^{n-1} k\sin(k\pi/n)}{\sin(\pi/2n)}. \tag{41}$$

The proof is now completed by using the well-known formula $6\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)$ together with the following identity (see, e.g., [4]),

$$\sum_{k=1}^{n-1} k\sin(k\alpha) = \frac{\sin(n\alpha)}{4\sin^2(\alpha/2)} - \frac{n\cos((2n-1)\alpha/2)}{2\sin(\alpha/2)}. \tag{42}$$

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